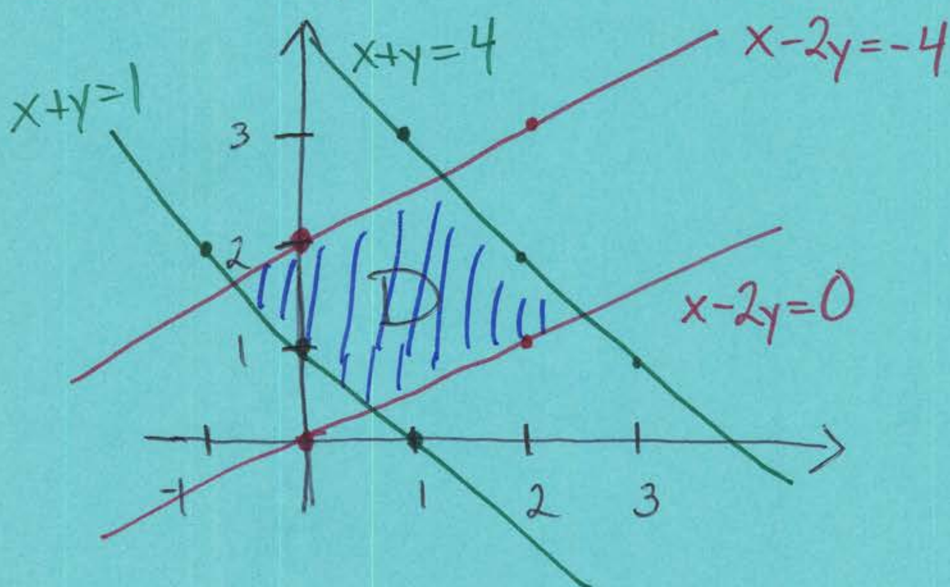
Lecture 30

15.10 - Change of variables in multiple integrals

Ex: Compute the integral of $3xy$ over the region bounded by the lines $x-2y=0$, $x-2y=-4$, $x+y=4$, and $x+y=1$.

Sol: First, as usual, sketch the region:



Computing $\iint_D 3xy \, dA$ using the methods at hand would require splitting this into 3 integrals with not-so-nice bounds. So, what do we do... ?

When our region was a disk, we switched to polar, and that made the integral easier, is there a change of variables which makes this easier?

Notice the edges show up in two groups:

$$\begin{cases} x-2y=0 \\ x-2y=-4 \end{cases} \quad \text{and} \quad \begin{cases} x+y=4 \\ x+y=1 \end{cases}$$

one with the expression $x-2y=\text{const.}$ & one with $x+y=\text{const.}$

If we let $u=x-2y$ and $v=x+y$, then in the uv-plane, the region is bounded by $u=0, u=-4, v=4, v=1$, a simple square! We need expressions for x & y in terms of u & v to rewrite the integrand in terms of u & v:

$$\begin{cases} u=x-2y & ① \\ v=x+y & ② \end{cases} \qquad ②-① \Rightarrow v-u=3y \Rightarrow y=\frac{1}{3}(v-u)$$

Plug into ②: $x=v-y=v-\frac{1}{3}(v-u)=\frac{1}{3}(2v+u)$

$$\begin{cases} x=\frac{1}{3}(u+2v) \\ y=\frac{1}{3}(-u+v) \end{cases}$$

Now, what about dA?

Sparing the proof, which can be found in the textbook, we have

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$. $\frac{\partial(x,y)}{\partial(u,v)}$ is called the <u>Jacobian</u> of the transformation

You can think of that notation as meaning "take the derivatives of these things (x and y) with respect to these things (u and v)."

So, in our case:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} - \left(\frac{-2}{9}\right) = \frac{3}{9} = \frac{1}{3}$$

Thus, $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \left|\frac{1}{3}\right| du\,dv = \frac{1}{3}\,du\,dv$

Now, $3xy = \cancel{3}\left(\cancel{\tfrac{1}{3}}(u+2v)\right)\left(\frac{1}{3}(-u+v)\right) = \frac{1}{3}\left(-u^2-2uv+uv+2v^2\right)$

$$= \frac{1}{3}\left(-u^2-uv+2v^2\right)$$

So,

$$\iint_D 3xy\,dA = \int_1^4 \int_{-4}^0 \frac{1}{3}(-u^2-uv+2v^2)\left(\frac{1}{3}\,du\,dv\right)$$

$$= \frac{1}{9}\int_1^4 \int_{-4}^0 (-u^2-uv+2v^2)\,du\,dv = \cdots = \frac{164}{9}$$

$\diamond$

Change of Variable Formula (2 variables)

Suppose $T(u,v) = (x(u,v), y(u,v)) = (g(u,v), h(u,v))$ is C^1 and sends the region S in the uv-plane to the region R in the xy-plane. If the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is nonzero at all points of S, $f(x,y)$ is continuous on R, and T is one-to-one on S (no two points map to the same image point) except maybe on the boundary of S, then:

$$\boxed{\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}.$$

T being C^1 means that g and h have continuous first partial derivatives.

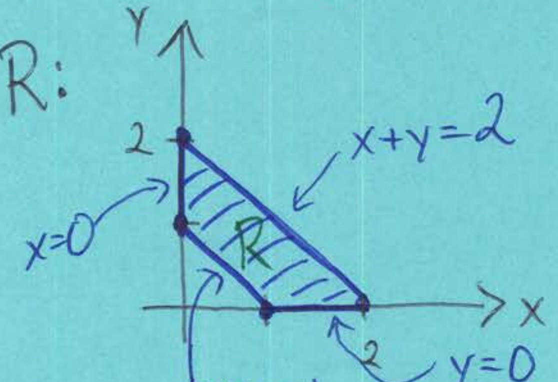
Change of coordinates isn't only useful for making regions easier, but also to make functions easier to integrate. This is akin to u-substitution.

<u>Ex</u>: Compute $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$ where R is the trapezoidal region with vertices $(1,0), (2,0), (0,2), (0,1)$.

<u>Sol</u>: $\cos\left(\frac{y-x}{y+x}\right)$ looks difficult to integrate, so let's try letting $u = y-x$ and $v = y+x$. Then

$\cos\left(\frac{y-x}{y+x}\right) = \cos\left(\frac{u}{v}\right)$, a bit nicer. Let's figure out the region of integration now:

R:

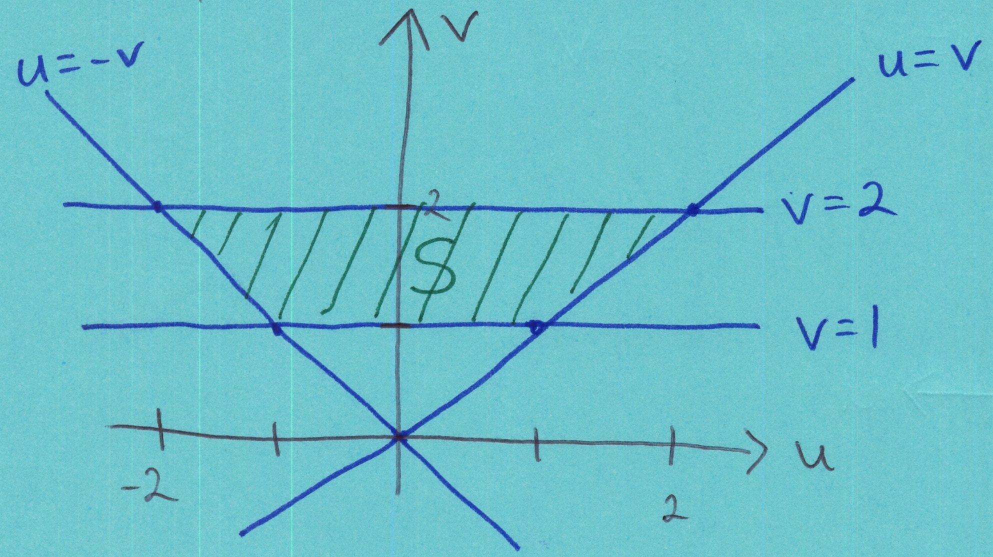


What region, S, in the uv-plane maps to this under the transformation $T(u,v) = (x(u,v), y(u,v))$?

xy-plane	uv-plane
$x+y=2$	$\underline{v=2}$ $\quad (v=x+y)$
$x+y=1$	$\underline{v=1}$
$x=0$	Plug this into eqn's for u&v: $\begin{cases} u=y-x=y \\ v=y+x=y \end{cases} \Rightarrow \underline{u=v}$
$y=0$	$\begin{cases} u=y-x=-x \\ v=y+x=x \end{cases} \Rightarrow \underline{u=-v}$

So, the region, S, in the uv-plane is bounded by:

$v=2,\ v=1,\ u=v,\ u=-v.$



This region is easiest to integrate over if we use horizontal slices (u first).

Finally, we need the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$. Since we didn't have to solve for x or y to find S or change

The function $\cos\left(\frac{y-x}{y+x}\right)$ into one depending on u and v (indeed, we chose u and v based on the integrand), we can use the following trick:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

(This has to do with the fact that for a square matrix, A, $\det(A^{-1}) = \frac{1}{\det(A)}$.)

So, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1 - 1 = -2$

$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{-1}{2}$

So,

$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \int_1^2 \int_{-v}^{v} \cos\left(\frac{u}{v}\right) \left|\frac{-1}{2}\right| du\,dv = \frac{1}{2}\int_1^2 v\sin\left(\frac{u}{v}\right)\Big|_{-v}^{v} dv$$

$$= \frac{1}{2}\int_1^2 \left(v\sin\frac{v}{v} - v\sin\frac{-v}{v}\right) dv = \frac{1}{2}\int_1^2 v\sin 1 - v\sin(-1)\,dv$$

$$= \int_1^2 v\sin(1)\,dv = \sin(1)\left(\frac{1}{2}v^2\Big|_1^2\right) = \sin(1)\left(2-\frac{1}{2}\right) = \frac{3\cdot\sin(1)}{2} \quad \diamond$$

We also have change of variables for triple integrals.

If $x=g(u,v,w)$, $y=h(u,v,w)$, $z=k(u,v,w)$, the Jacobian of this transformation is:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Under similar hypotheses as before

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

<u>Ex</u>: Show the Jacobian for the change to spherical coordinates is $\rho^2 \sin\varphi$.

<u>Sol</u>: $x=\rho\cos\theta\sin\varphi$, $y=\rho\sin\theta\sin\varphi$, $z=\rho\cos\varphi$.

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}$$

$$= \cos\theta\sin\varphi\left(-\rho^2\cos\theta\sin^2\varphi\right) + \rho\sin\theta\sin\varphi\left(-\rho\sin\theta\sin^2\varphi - \rho\sin\theta\cos^2\varphi\right)$$
$$+\rho\cos\theta\cos\varphi\left(-\rho\cos\theta\sin\varphi\cos\varphi\right)$$

$$= \underline{-\rho^2\sin\varphi(\cos^2\theta\sin^2\varphi)} - \underline{\rho^2\sin\varphi(\sin^2\theta\sin^2\varphi + \sin^2\theta\cos^2\varphi)}$$
$$\underline{-\rho^2\sin\varphi(\cos^2\theta\cos^2\varphi)} = -\rho^2\sin\varphi\cos^2\theta - \rho^2\sin\varphi\sin^2\theta = -\rho^2\sin\varphi \quad \diamond$$